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ASYMPTOTIC BEHAVIOR OF THE FAR STRESS FIELD IN THE PROBLEM OF CRACK GROWTH IN A DAMAGED MEDIUM UNDER CREEP CONDITIONS

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An asymptotic analysis of stress fields, creep-strain rates, and continuity in the vicinity of the tip of a crack that grows under creep conditions is performed with allowance for accumulation of dissipated damages. The configuration of a region of a fully damaged material adjacent to the crack edges and its tip is determined and studied. It is shown that the Hutchinson–Rice–Rosengren solution cannot be used as the boundary condition at an infinite point, and a new asymptotic representation of the far stress field, governing the geometry of the region of the fully damaged material, is obtained.

Key words: *stress field, crack, damage, asymptotic analysis.*

Introduction. An analysis of the stress–strain state of a cracked body with allowance for damage accumulation is of particular interest, because combinations of the stress-tensor components and the damage parameter are normally parts of the fracture criterion and, hence, determine the operation conditions of a structural element.

Much attention has been given recently to the coupled formulation of the problem of stress fields, strains, and continuity (damage); possible variants of the problem are the elasticity–damage, plasticity–damage, and creep–damage formulations [1–7]. The coupled formulation is dictated, on one hand, by the necessity of describing the effect of microdefects accumulated in a body with a macroscopic crack on the stress–strain state and, on the other hand, by the desire to take into account the reverse process, namely, variation in the stress–strain state due to formation and growth of microdefects.

It has been found [1–7] that, as a result of damages accumulated in a body with a macrocrack, the stress-field singularity in the vicinity of the crack tip predicted by the linear and nonlinear fracture mechanics is absent or substantially attenuated.

In the present paper, it is shown that accumulated damages affect not only the near stress field (in the vicinity of the crack tip) but also the far stress field (remote from the crack tip).

To study the stress–strain state in the vicinity of the crack tip, an approach used in constructing the boundary-layer theory [8] or the "microscope principle" [9] is commonly employed. These approaches imply investigations of the region in the vicinity of the crack tip; in this formulation, the crack is assumed to be semi-infinite and the true boundary conditions are replaced by conditions of asymptotic converging, e.g., the singular elastic solution for the case of a crack in an elastoplastic material under the assumption of a small-scale plastic flow [10, 11]. In this case, the plastic-flow region is said to be "completely controlled" by the singular elastic solution. A similar method for solving the problem of a crack growing in an elastoplastic material is used to formulate the boundary condition at an infinite point [12–14] and solve the problems of fracture mechanics in the coupled formulation. For example, the stresses and strains near the tip of antiplane-shear cracks and tensile cracks were studied in the coupled formulations (elasticity–damage and plasticity–damage) in [5] and [6, 7], respectively, under the assumption that the stress field in an immediate vicinity of the crack tip is distorted owing to damage accumulation, whereas it is completely

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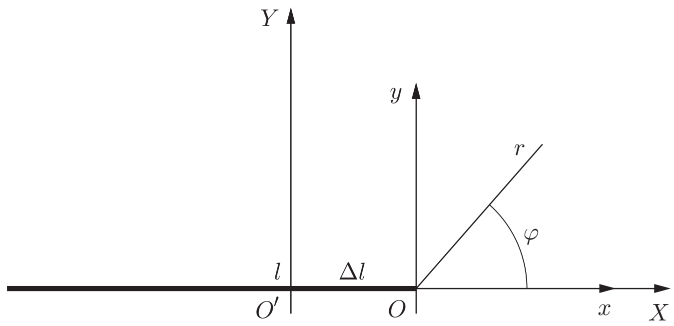

Fig. 1. Geometry of the growing crack tip: $XO'Y$ is the fixed coordinate system and xOy is the coordinate system attached to the growing crack tip.

determined by the singular elastic solution at a distance from the crack tip where the material remains undamaged. Thus, a hypothesis is used in which the region of accumulated dissipated damages is completely determined by the singular elastic solution.

A similar approach is used in the formulation of the boundary condition at an infinite point in an elastic nonlinear-viscous material [15, 16].

In the present paper, the constitutive relations $\dot\varepsilon_{ij} = (3B/2)(\sigma_e/\psi)^{n-1}s_{ij}/\psi$ based on the power law of the steady creep theory are considered.

In studying the continuity field far from the crack tip, we can assume that the continuity parameter tends to unity. In this case, the constitutive relations of the problem considered become identical with the power law of steady-state creep; hence, the boundary conditions can be formulated as conditions of asymptotic converging with the Hutchinson–Rice–Rosengren (HRR) solution [17, 18].

However, an asymptotic analysis of the kinetic equation shows that the HRR solution cannot be used as the boundary condition at an infinite point. Consequently, the effect of the damage accumulation process is also manifested in stress-field variation at distances from the crack tip, which are much greater than the characteristic linear size of the region of the fully damaged material modeled in the vicinity of the crack tip but still smaller than the crack length and the characteristic linear size of the body.

1. Formulation of the Problem. We consider a semi-infinite crack propagating in an infinite body (Fig. 1). Let the constitutive relations of the material be constructed on the basis of the Norton relation between the creep strain rates and stresses:

$$\dot\varepsilon_{ij} = \frac{3}{2}\,B\Big(\frac{\sigma_e}{\psi}\Big)^{n-1}\frac{s_{ij}}{\psi}. \tag{1.1}$$

Here $\dot\varepsilon_{ij}$ are the components of the creep strain rate tensor, B and n are the material constants, σ_e is the stress intensity [$\sigma_e^2 = 3(\sigma_{rr}-\sigma_{\varphi\varphi})^2/4+3\sigma_{r\varphi}^2$ for the plane strain state and $\sigma_e^2 = \sigma_{rr}^2+\sigma_{\varphi\varphi}^2-\sigma_{rr}\sigma_{\varphi\varphi}+3\sigma_{r\varphi}^2$ for the plane stress state, where σ_{ij} are the stress tensor components], ψ ($0 \leqslant \psi \leqslant 1$) is Kachanov's continuity parameter [19] ($\omega = 1-\psi$ ($0 \leqslant \omega \leqslant 1$) is Rabotnov's damage parameter [20]), and $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{kk}/3$ are the stress deviator components [$s_{rr} = -s_{\varphi\varphi} = (\sigma_{rr} - \sigma_{\varphi\varphi})/2$ for the plane strain state and $s_{rr} = (2\sigma_{rr} - \sigma_{\varphi\varphi})/3$ and $s_{\varphi\varphi} = (2\sigma_{\varphi\varphi} - \sigma_{rr})/3$ for the plane stress state].

We consider the stress fields, creep-strain rates, and scalar continuity parameter far from the tip of a mode I crack growing under conditions of the plane strain state or the plain stress state. The stress–strain state in the vicinity of the tip of the growing crack in a damaged material with constitutive relations of the type (1.1) has been the subject of numerous studies [1–4]. Astaf'ev et al. [1, 2] showed that the vicinity of the crack edges and its tip contains a region of the fully damaged material or (and) a zone where damages (micropores, microcracks, and microdefects) are intensely accumulated, which is sometimes called the process zone. For this reason, the traditional equations of mechanics of continuous media cannot be formulated in the vicinity of the tip of the growing crack. Therefore, we assume that a region of the fully damaged material in which all stress-tensor components and the continuity parameter vanish exists in the vicinity of the tip of the defect. We study the governing system of equations (equations of equilibrium, strain-compatibility condition, and kinetic equation of damage accumulation) at a significant distance from the tip of the defect to determine the asymptotic behavior of the far stress field and the configuration of the region of the fully damaged material.

Thus, it is necessary to find the solution of a system of equations written in polar coordinates with the pole located at the tip of a propagating crack (see Fig. 1), which includes:

— the equations of equilibrium

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\,\frac{\partial\sigma_{r\varphi}}{\partial\varphi}+\frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{r}=0,\qquad \frac{\partial\sigma_{r\varphi}}{\partial r}+\frac{1}{r}\,\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi}+2\,\frac{\sigma_{r\varphi}}{r}=0; \tag{1.2}$$

— the Cauchy relations between displacements and strains, i.e.,

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\qquad \varepsilon_{\varphi\varphi}=\frac{u_r}{r}+\frac{1}{r}\,\frac{\partial u_\varphi}{\partial\varphi},\qquad 2\varepsilon_{r\varphi}=\frac{1}{r}\,\frac{\partial u_r}{\partial\varphi}+\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}, \tag{1.3}$$

where ε_{ij} are the strain-tensor components and u_i are the displacements, the strain-compatibility condition formulated for the creep strain rates, namely,

$$2\,\frac{\partial}{\partial r}\Big(r\,\frac{\partial\dot\varepsilon_{r\varphi}}{\partial\varphi}\Big)=\frac{\partial^2\dot\varepsilon_{rr}}{\partial\varphi^2}-r\,\frac{\partial\dot\varepsilon_{rr}}{\partial r}+r\,\frac{\partial^2(r\dot\varepsilon_{\varphi\varphi})}{\partial r^2}; \tag{1.4}$$

— the kinetic equation that postulates the power law of damage accumulation

$$\frac{d\psi}{dt}=-A\Big(\frac{\sigma_{\rm eq}}{\psi}\Big)^m, \tag{1.5}$$

where A and m are the material constants, t is the time, and $\sigma_{\rm eq}=\alpha\sigma_1+\beta\sigma_e+(1-\alpha-\beta)\sigma_{kk}$ is the equivalent stress (σ_1 is the maximum principal stress and σ_{kk} is the hydrostatic stress; the constants α and β are determined experimentally). If the crack grows with a rate $v(t)$ in the x direction, the material derivative with respect to the time t is

$$\frac{d}{dt}=\frac{\partial}{\partial t}-v\,\frac{\partial}{\partial x}=\frac{\partial}{\partial t}-v\Big(\cos\varphi\,\frac{\partial}{\partial r}-\frac{\sin\varphi}{r}\,\frac{\partial}{\partial\varphi}\Big).$$

Confining ourselves to the case of steady growth of the crack, where the unknown quantities have no explicit dependence on time, we write the derivative with respect to time as

$$\frac{d}{dt}=-v\,\frac{\partial}{\partial x}=-v\Big(\cos\varphi\,\frac{\partial}{\partial r}-\frac{\sin\varphi}{r}\,\frac{\partial}{\partial\varphi}\Big).$$

In the case of steady growth of the crack, we have

$$\frac{\partial\psi}{\partial t}\equiv 0,$$

and the kinetic equation becomes

$$-v\Big(\cos\varphi\,\frac{\partial\psi}{\partial r}-\frac{\sin\varphi}{r}\,\frac{\partial\psi}{\partial\varphi}\Big)=-A\Big(\frac{\sigma_{\rm eq}}{\psi}\Big)^m. \tag{1.6}$$

The constitutive relations (1.1) are written as

$$\dot\varepsilon_{rr}=-\dot\varepsilon_{\varphi\varphi}=\frac{3}{4}\,B\Big(\frac{\sigma_e}{\psi}\Big)^{n-1}\frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{\psi},\qquad \dot\varepsilon_{r\varphi}=\frac{3}{2}\,B\Big(\frac{\sigma_e}{\psi}\Big)^{n-1}\frac{\sigma_{r\varphi}}{\psi} \tag{1.7}$$

for the plane strain state and

$$\dot\varepsilon_{rr}=\frac{1}{2}\,B\Big(\frac{\sigma_e}{\psi}\Big)^{n-1}\frac{2\sigma_{rr}-\sigma_{\varphi\varphi}}{\psi},\qquad \dot\varepsilon_{\varphi\varphi}=\frac{1}{2}\,B\Big(\frac{\sigma_e}{\psi}\Big)^{n-1}\frac{2\sigma_{\varphi\varphi}-\sigma_{rr}}{\psi},\qquad \dot\varepsilon_{r\varphi}=\frac{3}{2}\,B\Big(\frac{\sigma_e}{\psi}\Big)^{n-1}\frac{\sigma_{r\varphi}}{\psi} \tag{1.8}$$

for the plane stress state.

The traction-free boundary conditions at the crack edges have the form

$$\sigma_{\varphi\varphi}(r,\varphi=\pm\pi)=0,\qquad \sigma_{r\varphi}(r,\varphi=\pm\pi)=0. \tag{1.9}$$

The boundary condition at infinity is

$$\sigma_{ij}(r\to\infty,\varphi)\to\tilde C r^{s_0}\bar\sigma_{ij}(\varphi,n), \tag{1.10}$$

where the values of s_0 are determined in the course of solving the problem and $\bar\sigma_{ij}(\varphi,n)$ are functions to be determined.

It should be noted that the boundary condition (1.10) for the constitutive relations (1.1) can be formulated at an infinite point as

$$\sigma_{ij}(r \to \infty, \varphi) = \Big(\frac{C^*}{BI_n r}\Big)^{1/(n+1)} \bar\sigma_{ij}(\varphi, n), \tag{1.11}$$

because we have $\psi \equiv 1$ at infinity, and the two-term asymptotic expansion of the scalar continuity parameter at large distances from the crack tip is sought in the form

$$\psi(r,\varphi) = 1 - r^{\gamma_1} g^{(1)}(\varphi) + o(r^{\gamma_1}), \qquad \gamma_1 < 0, \quad r \to \infty. \tag{1.12}$$

In this case, the constitutive equations (1.1) are reduced to conventional relations of the power-law steady creep. However, as is shown below, the kinetic equation of damage accumulation (1.6) yields the relation $\gamma_1 = 1 - m/(n+1)$, and since $m = 0.7n$, then $\gamma_1 > 0$, which contradicts the condition $\gamma_1 < 0$ in (1.12). Consequently, the boundary condition (1.11) is used in a more general form (1.10). The quantity $\tilde C$ can be determined by solving the problem of a real structural element under certain loads with allowance for true boundary conditions.

The solution of the boundary-value problem (1.2)–(1.8) subject to the boundary conditions (1.9) and (1.10) is a function of the following variables and parameters of the problem: r, φ, A, m, v, $\tilde C$, B, and n.

A dimension analysis suggests that we can pass to the dimensionless functions

$$\sigma_{ij}(r,\varphi) = [\tilde C (v/A)^{s_0}]^{1/(s_0 m+1)} \tilde\sigma_{ij}(\tilde r, \varphi), \quad \dot\varepsilon_{ij}(r,\varphi) = 3B\tilde{\dot\varepsilon}_{ij}(\tilde r,\varphi)/2, \quad \psi(r,\varphi) = \tilde\psi(\tilde r,\varphi),$$

where $\tilde\sigma_{ij}$, $\tilde{\dot\varepsilon}_{ij}$, and $\tilde\psi$ are dimensionless functions of the dimensionless variables φ and $\tilde r = r/r_0$, where $r_0 = [\tilde C^{-m} v/A]^{1/(s_0 m+1)}$ (below, the tilde is omitted).

After introduction of the dimensionless quantities, the equations of equilibrium and the compatibility condition retain their form. The kinetic equation becomes

$$\cos\varphi \,\frac{\partial\psi}{\partial r} - \frac{\sin\varphi}{r}\,\frac{\partial\psi}{\partial\varphi} = \Big(\frac{\sigma_{\rm eq}}{\psi}\Big)^m. \tag{1.13}$$

The constitutive relations (1.7) and (1.8) are written in the dimensionless variables as

$$\dot\varepsilon_{rr} = -\dot\varepsilon_{\varphi\varphi} = \frac12\Big(\frac{\sigma_e}{\psi}\Big)^{n-1}\frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{\psi}, \qquad \dot\varepsilon_{r\varphi} = \Big(\frac{\sigma_e}{\psi}\Big)^{n-1}\frac{\sigma_{r\varphi}}{\psi} \tag{1.14}$$

for the plane strain state and

$$\dot\varepsilon_{rr} = \frac13\Big(\frac{\sigma_e}{\psi}\Big)^{n-1}\frac{2\sigma_{rr}-\sigma_{\varphi\varphi}}{\psi}, \quad \dot\varepsilon_{\varphi\varphi} = \frac13\Big(\frac{\sigma_e}{\psi}\Big)^{n-1}\frac{2\sigma_{\varphi\varphi}-\sigma_{rr}}{\psi}, \quad \dot\varepsilon_{r\varphi} = \Big(\frac{\sigma_e}{\psi}\Big)^{n-1}\frac{\sigma_{r\varphi}}{\psi} \tag{1.15}$$

for the plane stress state.

In the dimensionless variables, the boundary conditions at infinity acquire the form

$$\sigma_{ij}(r\to\infty,\varphi) \to r^{s_0}\bar\sigma_{ij}(\varphi,n). \tag{1.16}$$

Thus, we have to find the solution of system (1.2)–(1.4), (1.7), (1.8), (1.13) subject to the boundary conditions (1.16) and

$$\sigma_{\varphi\varphi}(r,\varphi=\pm\pi) = 0, \qquad \sigma_{r\varphi}(r,\varphi=\pm\pi) = 0.$$

2. Asymptotic Solution of the Problem. The stress-tensor components are expressed in terms of Airy's stress function $F(r,\varphi)$ as

$$\sigma_{\varphi\varphi} = \frac{\partial^2 F}{\partial r^2}, \qquad \sigma_{rr} = \Delta F - \sigma_{\varphi\varphi}, \qquad \sigma_{r\varphi} = -\frac{\partial}{\partial r}\Big(\frac1r\,\frac{\partial F}{\partial\varphi}\Big), \tag{2.1}$$

where $\Delta = \partial^2/\partial r^2 + (1/r)\,\partial/\partial r + (1/r^2)\,\partial^2/\partial\varphi^2$ is the Laplace operator.

We seek for the solution of the system formulated above in the form of the power expansions

$$\begin{gathered} F(r,\varphi) = r^{\lambda_0} f^{(0)}(\varphi) + r^{\lambda_1} f^{(1)}(\varphi) + o(r^{\lambda_1}) \qquad (\lambda_j < 0), \\ \psi(r,\varphi) = 1 - r^{\gamma_1} g^{(1)}(\varphi) + o(r^{\gamma_1}) \qquad (\gamma_j < 0) \end{gathered} \tag{2.2}$$

as $r \to \infty$, moving from an infinite point to the vicinity of the crack tip; we have to determine the eigenfunctions $f^{(j)}(\varphi)$ and $g^{(j)}(\varphi)$ and the eigenvalues λ_j and γ_j.

By virtue of Eqs. (2.1) and (2.2), the two-term asymptotic expansions of the stress-tensor components, stress intensity, and equivalent stress are given by the equalities

$$\sigma_{rr}(r,\varphi) = r^{s_0}[\lambda_0 f^{(0)} + (f^{(0)})''] + r^{s_1}[\lambda_1 f^{(1)} + (f^{(1)})''],$$
$$\sigma_{\varphi\varphi}(r,\varphi) = r^{s_0}\lambda_0(\lambda_0-1)f^{(0)} + r^{s_1}\lambda_1(\lambda_1-1)f^{(1)},$$
$$\sigma_{r\varphi}(r,\varphi) = r^{s_0}(1-\lambda_0)(f^{(0)})' + r^{s_1}(1-\lambda_1)(f^{(1)})'$$

or

$$\sigma_{rr}(r,\varphi) = r^{s_0} f_{rr}^{(0)}(\varphi) + r^{s_1} f_{rr}^{(1)}(\varphi), \qquad \sigma_{\varphi\varphi}(r,\varphi) = r^{s_0} f_{\varphi\varphi}^{(0)}(\varphi) + r^{s_1} f_{\varphi\varphi}^{(1)}(\varphi),$$
$$\sigma_{r\varphi}(r,\varphi) = r^{s_0} f_{r\varphi}^{(0)}(\varphi) + r^{s_1} f_{r\varphi}^{(1)}(\varphi), \qquad s_0 = \lambda_0 - 2, \quad s_1 = \lambda_1 - 2, \tag{2.3}$$

where

$$f_{rr}^{(j)}(\varphi) = \lambda_j f^{(j)} + (f^{(j)})'', \quad f_{\varphi\varphi}^{(j)}(\varphi) = \lambda_j(\lambda_j-1)f^{(j)}, \quad f_{r\varphi}^{(j)} = (1-\lambda_j)(f^{(j)})'; \tag{2.4}$$

$$\sigma_e(r,\varphi) = r^{s_0}\sigma_e^{(0)}(\varphi) + r^{s_1}\sigma_e^{(1)}(\varphi), \qquad \sigma_{\text{eq}}(r,\varphi) = r^{s_0}\sigma_{\text{eq}}^{(0)}(\varphi) + r^{s_1}\sigma_{\text{eq}}^{(1)}(\varphi).$$

For the plane strain state, we have

$$(\sigma_e^{(0)})^2 = (3/4)\{\lambda_0^2(\lambda_0-2)^2(f^{(0)})^2 + 4(\lambda_0-1)^2[(f^{(0)})']^2 - 2\lambda_0 s_0 f^{(0)}(f^{(0)})'' + [(f^{(0)})'']^2\},$$

$$\sigma_e^{(1)} = \{[\lambda_0(2-\lambda_0)f^{(0)} - (f^{(0)})''][\lambda_1(2-\lambda_1)f^{(1)} - (f^{(1)})''] + 4(1-\lambda_0)(1-\lambda_1)(f^{(0)})'(f^{(1)})'\}/\sigma_e^{(0)},$$

and for the plane stress state, we have

$$(\sigma_e^{(0)})^2 = \lambda_0^2(\lambda_0^2 - 3\lambda + 3)(f^{(0)})^2 + 3(\lambda_0-1)^2[(f^{(0)})']^2 + \lambda_0(3-\lambda_0)f^{(0)}(f^{(0)})'' + [(f^{(0)})'']^2,$$

$$\sigma_e^{(1)} = \{(\lambda_0 f^{(0)} + (f^{(0)})'')(\lambda_1 f^{(1)} + (f^{(1)})'') + \lambda_0(\lambda_0-1)\lambda_1(\lambda_1-1)f^{(0)}f^{(1)}$$
$$- [(\lambda_0 f^{(0)} + (f^{(0)})'')\lambda_1(\lambda_1-1)f^{(1)} + (\lambda_1 f^{(1)} + (f^{(1)})'')\lambda_0(\lambda_0-1)f^{(0)}] + 3(1-\lambda_0)(1-\lambda_1)(f^{(0)})'(f^{(1)})'\}/\sigma_e^{(0)},$$

$$\sigma_{\text{eq}}^{(0)}(\varphi) = \alpha\sigma_1^{(0)}(\varphi) + \beta\sigma_e^{(0)}(\varphi) + (1-\alpha-\beta)\sigma_{kk}^{(0)}(\varphi).$$

The constant γ_1 is determined by an asymptotic analysis of the kinetic equation (1.13). Substituting the asymptotic expansions (2.2) into the kinetic equation (1.13), we obtain

$$\gamma_1 \cos\varphi\, r^{\gamma_1-1} g^{(1)}(\varphi) - \sin\varphi\, r^{\gamma_1-1}(g^{(1)}(\varphi))' = -r^{(\lambda_0-2)m}(\sigma_{\text{eq}}^{(0)}(\varphi))^m.$$

Therefore, if we accept the hypothesis of identical orders of the quantities as $r \to \infty$, which enter the left and right sides of the last equation, the equalities $\gamma_1 - 1 = (\lambda_0 - 2)m$ or $\gamma_1 = 1 + s_0 m$ and

$$\sin\varphi\,(g^{(1)}(\varphi))' - \gamma_1\cos\varphi\, g^{(1)}(\varphi) = (\sigma_{\text{eq}}^{(0)}(\varphi))^m \tag{2.5}$$

are valid.

Given the two-term asymptotic expansions of the stress-tensor components (2.3) and the continuity parameter [the second expression in (2.2)], one can obtain two-term asymptotic expansions of the components of the creep strain-rate tensor. Substitution of Eq. (2.3) and the second asymptotic expansion from (2.2) into (1.14) and (1.15) yields the following two-term asymptotic expansions of the creep strain rates (as $r \to \infty$):

— for the plane strain state, we have

$$\dot\varepsilon_{rr}(r,\varphi) = -\dot\varepsilon_{\varphi\varphi}(r,\varphi) = r^{s_0 n}\varepsilon_{rr}^{(0)}(\varphi) + r^{s_0 n + s_1 - s_0}\varepsilon_{rr}^{(1)}(\varphi),$$
$$\dot\varepsilon_{r\varphi} = r^{s_0 n}\varepsilon_{r\varphi}^{(0)}(\varphi) + r^{s_0 n + s_1 - s_0}\varepsilon_{r\varphi}^{(1)}(\varphi), \tag{2.6}$$

where

$$\varepsilon_{rr}^{(0)}(\varphi) = (1/2)(\sigma_e^{(0)})^{n-1}(f_{rr}^{(0)} - f_{\varphi\varphi}^{(0)}), \qquad \varepsilon_{r\varphi}^{(0)}(\varphi) = (\sigma_e^{(0)})^{n-1} f_{r\varphi}^{(0)}; \tag{2.7}$$

$$\varepsilon_{rr}^{(1)}(\varphi) = (1/2)(\sigma_e^{(0)})^{n-1}\{f_{rr}^{(1)} - f_{\varphi\varphi}^{(1)} + (f_{rr}^{(0)} - f_{\varphi\varphi}^{(0)})[ng^{(1)} + (n-1)\sigma_e^{(1)}/\sigma_e^{(0)}]\};$$
$$\varepsilon_{r\varphi}^{(1)}(\varphi) = (\sigma_e^{(0)})^{n-1}\{f_{r\varphi}^{(1)} + f_{r\varphi}^{(0)}[ng^{(1)} + (n-1)\sigma_e^{(1)}/\sigma_e^{(0)}]\}; \tag{2.8}$$

— for the plane stress state, we have

$$\begin{aligned}\dot{\varepsilon}_{rr}(r,\varphi) &= r^{s_0 n}\varepsilon_{rr}^{(0)}(\varphi) + r^{s_0 n + s_1 - s_0}\varepsilon_{rr}^{(1)}(\varphi),\\ \dot{\varepsilon}_{\varphi\varphi}(r,\varphi) &= r^{s_0 n}\varepsilon_{\varphi\varphi}^{(0)}(\varphi) + r^{s_0 n + s_1 - s_0}\varepsilon_{\varphi\varphi}^{(1)}(\varphi),\\ \dot{\varepsilon}_{r\varphi}(r,\varphi) &= r^{s_0 n}\varepsilon_{r\varphi}^{(0)}(\varphi) + r^{s_0 n + s_1 - s_0}\varepsilon_{r\varphi}^{(1)}(\varphi),\end{aligned} \tag{2.9}$$

where

$$\begin{gathered}\varepsilon_{rr}^{(0)}(\varphi) = (1/3)(\sigma_e^{(0)})^{n-1}(2f_{rr}^{(0)} - f_{\varphi\varphi}^{(0)}), \qquad \varepsilon_{\varphi\varphi}^{(0)}(\varphi) = (1/3)(\sigma_e^{(0)})^{n-1}(2f_{\varphi\varphi}^{(0)} - f_{rr}^{(0)}),\\ \varepsilon_{r\varphi}^{(0)}(\varphi) = (\sigma_e^{(0)})^{n-1}f_{r\varphi}^{(0)};\end{gathered} \tag{2.10}$$

$$\begin{gathered}\varepsilon_{rr}^{(1)}(\varphi) = (1/3)(\sigma_e^{(0)})^{n-1}\{2f_{rr}^{(1)} - f_{\varphi\varphi}^{(1)} + (2f_{rr}^{(0)} - f_{\varphi\varphi}^{(0)})[ng^{(1)} + (n-1)\sigma_e^{(1)}/\sigma_e^{(0)}]\},\\ \varepsilon_{\varphi\varphi}^{(1)}(\varphi) = (1/3)(\sigma_e^{(0)})^{n-1}\{2f_{\varphi\varphi}^{(1)} - f_{rr}^{(1)} + (2f_{\varphi\varphi}^{(0)} - f_{rr}^{(0)})[ng^{(1)} + (n-1)\sigma_e^{(1)}/\sigma_e^{(0)}]\},\\ \varepsilon_{r\varphi}^{(1)}(\varphi) = (\sigma_e^{(0)})^{n-1}\{f_{r\varphi}^{(1)} + f_{r\varphi}^{(0)}[ng^{(1)} + (n-1)\sigma_e^{(1)}/\sigma_e^{(0)}]\}.\end{gathered} \tag{2.11}$$

The second term of the asymptotic expansion of the creep strain rates (2.8) and (2.11) is derived under the assumption that $s_1 = s_0 + \gamma_1$. This equality is obtained by comparing the orders of terms at $r^{s_1 - s_0}$ and r^{γ_1} that enter the two-term asymptotic expansion of the strain rates. Only if these orders coincide, one can construct as many terms of the asymptotic expansion as desired. At each step of the process, one obtains an ordinary differential equation for a new unknown function (either $f^{(k)}$ or $g^{(k)}$) and no "non-balanced" terms appear in the compatibility condition. Thus, the exponents in the asymptotic expansion of the stress-tensor components s_k are determined as follows: the eigenvalue s_0 is calculated numerically (the procedure is described below) and s_k ($k > 0$) is found by an asymptotic analysis of the $(k+1)$-term expansion of the creep strain rates. The exponents γ_k are found by an asymptotic analysis of the kinetic equation of damage accumulation.

It should be noted that the leading term in the asymptotic expansions of the components of the creep strain rate tensor (2.7) and (2.10) is completely determined by the function $f^{(0)}(\varphi)$ by virtue of (2.4), which allows one to determine the unknown function $f^{(0)}(\varphi)$ from the strain-compatibility condition without determining the function $g^{(1)}(\varphi)$. Thus, the initially coupled problem becomes "uncoupled." Indeed, substituting (2.6) for the plane strain state and (2.9) for the plane stress state into the strain-compatibility condition and equating the coefficients of identical powers of r, we obtain two ordinary differential equations

$$2(s_0 n + 1)\,\frac{\partial \tilde{\varepsilon}_{r\varphi}^{(0)}}{\partial \varphi} = \frac{\partial^2 \tilde{\varepsilon}_{rr}^{(0)}}{\partial \varphi^2} - s_0 n(s_0 n + 2)\tilde{\varepsilon}_{rr}^{(0)}; \tag{2.12}$$

$$2(s_0 n + s_1 - s_0 + 1)\,\frac{\partial \tilde{\varepsilon}_{R\varphi}^{(1)}}{\partial \varphi} = \frac{\partial^2 \tilde{\varepsilon}_{rr}^{(1)}}{\partial \varphi^2} - (s_0 n + s_1 - s_0)(s_0 n + s_1 - s_0 + 2)\tilde{\varepsilon}_{rr}^{(1)} \tag{2.13}$$

for the plane strain state and

$$2(s_0 n + 1)\,\frac{\partial \tilde{\varepsilon}_{r\varphi}^{(0)}}{\partial \varphi} = \frac{\partial^2 \tilde{\varepsilon}_{rr}^{(0)}}{\partial \varphi^2} - s_0 n\tilde{\varepsilon}_{rr}^{(0)} + (s_0 n + 1)s_0 n\tilde{\varepsilon}_{\varphi\varphi}^{(0)}; \tag{2.14}$$

$$2(s_0 n + s_1 - s_0 + 1)\,\frac{\partial \tilde{\varepsilon}_{r\varphi}^{(1)}}{\partial \varphi} = \frac{\partial^2 \tilde{\varepsilon}_{rr}^{(1)}}{\partial \varphi^2} - (s_0 n + s_1 - s_0)\tilde{\varepsilon}_{rr}^{(1)} + (s_0 n + s_1 - s_0 + 1)sn\tilde{\varepsilon}_{\varphi\varphi}^{(1)} \tag{2.15}$$

for the plane stress state.

Equations (2.12) and (2.14) are the fourth-order nonlinear ordinary differential equations with respect to the function $f^{(0)}(\varphi)$. Therefore, the function $f^{(0)}(\varphi)$ is determined first, and then Eq. (2.5) is studied as an inhomogeneous ordinary differential equation with respect to the function $g^{(1)}(\varphi)$ whose right side is determined by the function $f^{(0)}(\varphi)$. Once the function $g^{(1)}(\varphi)$ is determined, one can obtain the numerical solution of the ordinary differential equation (2.13) [or (2.15)], which is the fourth-order linear ordinary differential equation with respect to the function $f^{(1)}(\varphi)$ [this equation contains the functions $f^{(0)}(\varphi)$ and $g^{(1)}(\varphi)$, which are already known functions at this step of constructing the asymptotic expansions]. The sequence of these operations can be continued. Thus, an algorithm for "decoupling" the system of equations of the coupled problem is proposed in the present paper.

At each step, one has to construct either the function $f^{(j)}(\varphi)$ after determining the functions $f^{(0)}, \dots, f^{(j-1)}$ and $g^{(1)}, \dots, g^{(j)}$ or the function $g^{(j)}(\varphi)$ after determining the functions $f^{(0)}, \dots, f^{(j-1)}$ and $g^{(1)}, \dots, g^{(j-1)}$ $(j \geq 1)$.

The present paper is aimed at determining a one-term expansion of Airy's stress function and a two-term expansion of the continuity parameter, which allows one to estimate the configuration of the region of the fully damaged material enclosing the tip of the crack and adjacent to its edges.

Substituting (2.7) into (2.12) and taking into account (2.4), we obtain the nonlinear ordinary differential equation with respect to the function $f^0(\varphi)$

$$
\begin{gathered}
(f^{(0)})^{IV} N(\varphi) = 4(s_0 n + 1)(1 - \lambda_0)[(n-1)K(\varphi)(f^{(0)})' + h^2 (f^{(0)})'']\\
+ h^2[(s_0\lambda_0 + s_0 n(s_0 n + 2))(f^{(0)})'' - s_0 n(s_0 n + 2)(2 - \lambda_0)\lambda_0 f^{(0)}]\\
- (n-1)(n-3)(K(\varphi)/h)^2[(f^{(0)})'' - s_0\lambda_0 f^{(0)}]\\
- 2(n-1)K(\varphi)[(f^{(0)})''' - s_0\lambda_0 (f^{(0)})'] - (n-1)M(\varphi)[(f^{(0)})'' - s_0\lambda_0 f^{(0)}], \qquad (2.16)
\end{gathered}
$$

where

$$
\begin{gathered}
K(\varphi) = [(f^{(0)})'' - s_0\lambda_0 f^{(0)}][(f^{(0)})''' - s_0\lambda_0(f^{(0)})'] + 4(1-\lambda_0)^2 (f^{(0)})'(f^{(0)})'',\\
M(\varphi) = -[(f^{(0)})'' - s_0\lambda_0 f^{(0)}]\lambda_0 s_0 (f^{(0)})'' + [-s_0\lambda_0 f^{(0)} + (f^{(0)})'']^2 + 4(1-\lambda_0)^2 (f^{(0)})'(f^{(0)})'',\\
N(\varphi) = n[(f^{(0)})'' - s_0\lambda_0 f^{(0)}(f^{(0)})'']^2 + 4(1-\lambda_0)^2[(f^{(0)})']^2,\\
h(\varphi) = \sqrt{[(f^{(0)})'' - \lambda_0 s_0 f^{(0)}]^2 + 4(1-\lambda_0)^2((f^{(0)})')^2}
\end{gathered}
$$

for the plane strain state. Substitution of (2.10) into (2.14) with allowance for (2.4) yields the nonlinear ordinary differential equation with respect to the function $f^0(\varphi)$

$$
\begin{gathered}
(f^{(0)})^{IV} N(\varphi) = 6(s_0 n + 1)(1 - \lambda_0)[(n-1)K(\varphi) + h^2 (f^{(0)})'']\\
- [(n-3)(K(\varphi)/h)^2 + (n-1)M(\varphi)][\lambda_0(3-\lambda_0)f^{(0)} + 2(f^{(0)})'']\\
- 2(n-1)K(\varphi)[\lambda_0(3-\lambda_0)(f^{(0)})' + 2(f^{(0)})'''] -\\
- h^2\lambda_0 s_0 n((s_0 n + 1)(2\lambda_0 - 3) - 3 + \lambda_0)f^{(0)} - h^2(\lambda_0(3-\lambda_0) - s_0 n(s_0 n + 1))(f^{(0)})'', \qquad (2.17)
\end{gathered}
$$

where

$$
\begin{gathered}
K(\varphi) = (\lambda_0 f^{(0)} + (f^{(0)})'')(\lambda_0 (f^{(0)})' + (f^{(0)})''') + \lambda_0^2(\lambda_0 - 1)^2 f^{(0)}(f^{(0)})'\\
+ 3(1-\lambda_0)^2(f^{(0)})'(f^{(0)})'' - \lambda_0(1-\lambda_0)(\lambda_0(f^{(0)})' + (f^{(0)})''')f^{(0)}/2 - \lambda_0(1-\lambda_0)(\lambda_0 f^{(0)} + (f^{(0)})'')(f^{(0)})'/2;\\
M(\varphi) = (\lambda_0(f^{(0)})' + (f^{(0)})''')^2 + \lambda_0(3-\lambda_0)(\lambda_0 f^{(0)} + (f^{(0)})'')(f^{(0)})''\\
+ \lambda_0^2(1-\lambda_0)^2((f^{(0)})')^2 + \lambda_0^2(\lambda_0 - 1)(2\lambda_0 - 3)f^{(0)}(f^{(0)})''/2 + 3(1-\lambda_0)^2((f^{(0)})'')^2\\
+ 3(1-\lambda_0)^2(f^{(0)})'(f^{(0)})''' - \lambda_0(\lambda_0 - 1)(\lambda_0(f^{(0)})' + (f^{(0)})''')(f^{(0)})';\\
N(\varphi) = (n-1)(\lambda_0(3-\lambda_0)f^{(0)} + 2(f^{(0)})'')^2/2 + 2h^2;\\
h = [(\lambda_0 f^{(0)} + (f^{(0)})'')^2 + \lambda_0^2(\lambda_0 - 1)^2(f^{(0)})^2 - (\lambda_0 f^{(0)} + (f^{(0)})'')\lambda_0(\lambda_0 - 1)f^{(0)} - 3(1-\lambda_0)^2((f^{(0)})')^2]^{1/2}
\end{gathered}
$$

for the plane stress state. The solution of the equations obtained should satisfy the traction-free boundary conditions at the crack edges

$$
f^{(0)}(\pi) = 0, \qquad (f^{(0)})'(\pi) = 0 \qquad (2.18)
$$

and the conditions of symmetry on the crack continuation

$$
(f^{(0)})'(0) = 0, \qquad (f^{(0)})'''(0) = 0. \qquad (2.19)
$$

TABLE 1

n	s_0	$f''(0)$
1	-1.5	-0.75
2	-1.0	-0.5
3	-0.7716	-0.4372
4	-0.6684	-0.4092
5	-0.6179	-0.3985
6	-0.5901	-0.3950
7	-0.5732	-0.3943
8	-0.5621	-0.3948
9	-0.5543	-0.3958

TABLE 2

n	s_0	$f''(0)$
1	-1.5	-0.75
2	-1.1540	-0.5686
3	-1.0	-0.5
4	-0.9133	-0.4658
5	-0.8580	-0.4428
6	-0.8197	-0.4261
7	-0.7919	-0.4134
8	-0.7708	-0.4035
9	-0.7543	-0.3955

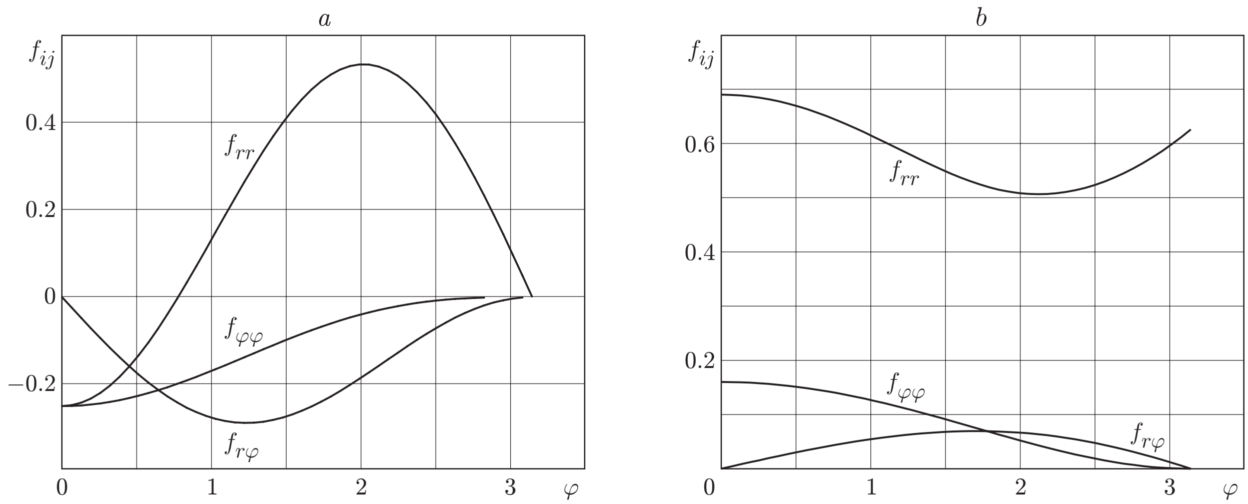

Fig. 2. Stress-tensor components as functions of the polar angle for the tensile crack in the plane stress state: (a) $n = 1$ and $m = 1$; (b) $n = 5$ and $m = 0.7n$.

To solve numerically Eqs. (2.16) and (2.17), we use the fifth-order Runge–Kutta–Fehlberg method combined with the shooting method. Equations (2.16) and (2.17) subject to the boundary conditions (2.18) and (2.19) are reduced to the Cauchy problem. For this purpose, the boundary conditions for $\varphi = \pi$ are replaced by the initial conditions $f^{(0)}(0) = c_1$ and $(f^{(0)})''(0) = c_2$ for $\varphi = 0$. Since Eqs. (2.16) and (2.17) are homogeneous, we can use the normalization condition $f^{(0)}(0) = 1$. Thus, the initial conditions are written as

$$f^{(0)}(0) = 1, \quad (f^{(0)})'(0) = 0, \quad (f^{(0)})''(0) = c_2, \quad (f^{(0)})'''(0) = 0.$$

In solving numerically system (2.16), (2.17), we determine the eigenvalues s_0 and the constant c_2 for various n such that the boundary conditions are satisfied for $\varphi = \pi$: $f^{(0)}(\pi) = 0$ and $(f^{(0)})'(\pi) = 0$. To determine the constants s_0 and c_2, we verify the conditions $(f^{(0)})^2(\pi) + ((f^{(0)})'(\pi))^2 \le \varepsilon$, where $\varepsilon = 10^{-5}$. The eigenvalues s_0 and the second derivative of the function $f^{(0)}(\varphi)$ on the continuation of the crack line $\varphi = 0$ are summarized in Tables 1 and 2 for the plane strain state and the plane stress state, respectively. Figures 2a and 2b show the stress-tensor components as functions of the polar angle for the tensile crack in the plane stress state for $n = 1$ and 5, respectively. For other values of n, the dependences of the stress-tensor components on the polar angle are similar to those shown in Fig. 2b except for the cases of $n = 2$ for the plane strain state and $n = 3$ for the plane stress state, where $\lambda_0 = 1$ and, hence, by virtue of (2.3), $\sigma_{ij} = \text{const}$ for all i and j.

Substituting the asymptotic expansion (2.2) into the kinetic equation (1.11), we obtain the differential equation for steady crack growth

$$\sin\varphi (g^{(1)}(\varphi))' - \gamma_1 \cos\varphi g^{(1)}(\varphi) = (\sigma_{\text{eq}}^{(0)}(\varphi))^m, \tag{2.20}$$

where $\gamma_1 = 1 + s_0 m$. The exponents γ_1 in the expansion of the scalar continuity parameter are listed in Tables 3 and 4 for the plane strain state and the plane stress state, respectively.

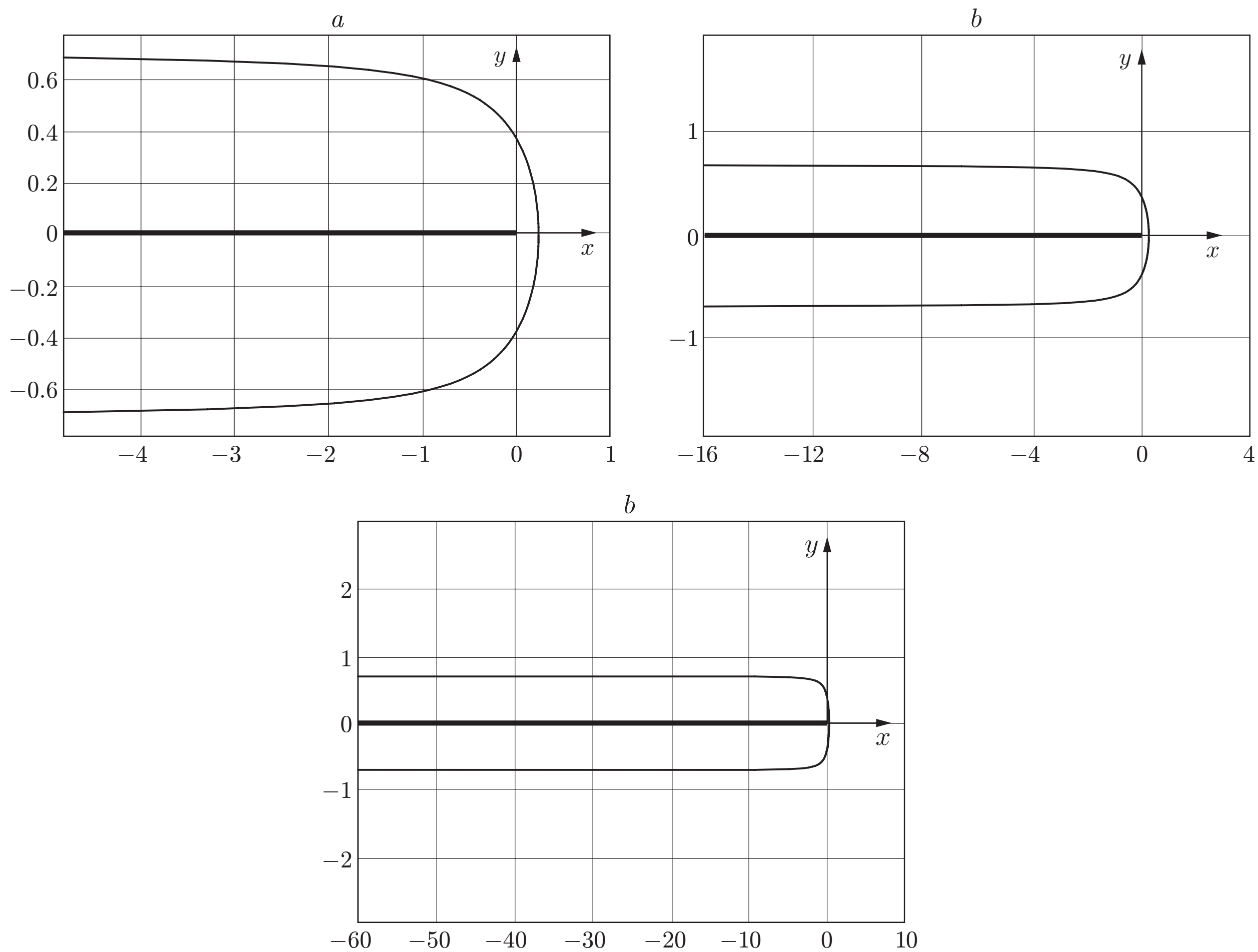

Fig. 3. Geometry of the region of the fully damaged material for $n = 3$ and $m = 0.7n$: (a) for an observation point located in the vicinity of the crack tip; (b, c) for observation points remote from the crack tip.

TABLE 3

n	m	γ_1
1	1	−0.5
2	$0.7n$	−0.4
3	$0.7n$	−0.6205
4	$0.7n$	−0.8717
5	$0.7n$	−1.1626
6	$0.7n$	−1.4785
7	$0.7n$	−1.8089
8	$0.7n$	−2.1479
9	$0.7n$	−2.4925

TABLE 4

n	m	γ_1
1	1	−0.5
2	$0.7n$	−0.6156
3	$0.7n$	−1.1
4	$0.7n$	−1.5574
5	$0.7n$	−2.0030
6	$0.7n$	−2.4431
7	$0.7n$	−2.8805
8	$0.7n$	−3.3167
9	$0.7n$	−3.7532

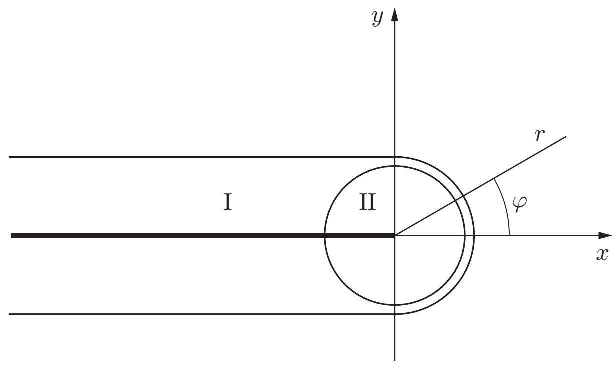

Fig. 4. Configuration of the region of the fully damaged material (I) and the zone of domination of the HRR solution (II).

The two-term asymptotic expansion of the continuity parameter allows one to estimate the shape and size of the region of the fully damaged material. Using the above-constructed dependences of the stress-tensor components on the polar angle, one obtains the function $g^{(1)}(\varphi)$ by solving numerically the ordinary differential equation (2.20) under the condition that its solution is regular for $\varphi = 0$:

$$g^{(1)}(0) = -(\sigma_{\rm eq}^{(0)}(0))^m/\gamma_1.$$

Since the continuity parameter vanishes on the boundary of this zone, $\psi(r, \varphi) = 1 - r^{\gamma_1} g^{(1)}(\varphi) = 0$, the equation of the boundary of the region of the fully damaged material has the form

$$r(\varphi) = [g^{(1)}(\varphi)]^{-1/\gamma_1}.$$

The configuration of the region of the fully damaged material is shown in Fig. 3 for observation points located at different distances from the tip of the growing crack.

Thus, it has been found that the HRR asymptotic solution, which can be regarded as the classical asymptotic representation of the stress-tensor components in the vicinity of the crack tip for the power relation between the components of stresses and strains (or strain rates), cannot be used as a boundary condition at an infinite point in the problem of a semi-infinite crack growing in a damaged medium for constitutive relations of the type considered. Failure to formulate the boundary condition at an infinite point as the requirement of asymptotic converging of the desired solution and the HRR solution can be explained by the fact that the region of the fully damaged material is much greater than the zone where the HRR solution dominates, so that the zone where the HRR solution holds is covered partly or completely by the region of the fully damaged material and, hence, the geometry of the latter cannot be governed by the HRR asymptotic solution (Fig. 4).

It is worth noting that the eigenvalue spectrum of this problem and the character of the singularity of the stress-tensor components at the crack tip were studied in [21], where the eigenvalues were determined only for some exponents in the steady-state creep power law ($n = 1$, 3, and 5). The eigenvalues obtained in the present paper coincide with those in [21] for all exponents n that are of practical importance (see Tables 1–4).

Conclusions. A comparatively simple approach proposed in the present paper allows one to determine the geometry of the region of the fully damaged material in the vicinity of the tip and edges of a crack. It should be noted that it is common practice to define the boundary of the region of the fully damaged material *a priori* on the basis of, e.g., experimental observations [3], where the boundary of the region ahead of the crack tip is described by an elliptic arc and additionally determined by two straight lines parallel to the crack edges beyond the crack tip. In contrast to the approach of [3], a unified relation $r = r(\varphi)$ is obtained in the present paper for determining the boundary of the fully damaged material.

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